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INFINITE HIGHLY ARC TRANSITIVE DIGRAPHS AND UNIVERSAL COVERING DIGRAPHS

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A digraph (that is a directed graph) is said to be highly arc transitive if its automorphism group is transitive on the set of s-arcs for each $s \geq 0$. Several new constructions are given of infinite highly arc transitive digraphs. In particular, for Δ a connected, 1-arc transitive, bipartite digraph, a highly arc transitive digraph $DL(\Delta)$ is constructed and is shown to be a covering digraph for every digraph in a certain class $\mathcal{D}(\Delta)$ of connected digraphs. Moreover, if Δ is locally finite, then $DL(\Delta)$ is a universal covering digraph for $\mathcal{D}(\Delta)$. Further constructions of infinite highly arc transitive digraphs are given.

1. Introduction

A *digraph* D consists of a set VD of vertices and a set $ED \subseteq VD \times VD$ of edges. In this paper it will be assumed that a digraph has no loops (except in Remark 4.4(b)), that is, if $(x,y) \in ED$, then $x \neq y$. For a non-negative integer s, an s-*arc* in a digraph D is a sequence $(x_0,\dots,x_s)$ of $s+1$ vertices of D such that, for each i $(0 \leq i < s)$, the pair $(x_i,x_{i+1}) \in ED$ and, for each i $(1 \leq i \leq s-1)$, $x_{i-1} \neq x_{i+1}$. A digraph D is said to be s-*arc transitive* if its automorphism group is transitive on the set of s-arcs, and D is said to be *highly arc transitive* if it is s-arc transitive for all finite $s \geq 0$. (In particular, a highly arc transitive digraph is vertex transitive.) For example, any directed tree with constant in- and out-valencies is highly arc transitive. In this paper, we attempt to characterise highly arc transitive digraphs in several ways. In particular, some new constructions are given for infinite highly arc transitive digraphs. One of these constructions is shown to produce digraphs which are universal covering digraphs for certain classes of connected 1-arc transitive digraphs. Also, certain properties of some highly arc transitive digraphs, suggested by these constructions, are studied.

Let D be a digraph. For a vertex x, we let $D(x) = \{y|(x,y) \in ED\}$ and $D^*(x) = \{y|(y,x) \in ED\}$. An *alternating walk* in D is a sequence $(x_1,\dots,x_n)$ of vertices of D such that either (x_{2i-1},x_{2i}) and (x_{2i+1},x_{2i}) are edges for all $i<n$, or

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(x_{2i}, x_{2i-1}) and (x_{2i}, x_{2i+1}) are edges for all $i < n$. If e and e' are edges in D and there is an alternating walk $(x_1, \ldots, x_n)$ such that (x_1, x_2) is e and either (x_{n-1}, x_n) or (x_n, x_{n-1}) is e', then e' is said to be *reachable* from e by an alternating walk; this is denoted by $e\mathcal{A}e'$. Clearly the relation $\mathcal{A}$ of reachability is an equivalence relation; the equivalence class containing e will be denoted by $\mathcal{A}(e)$. Let $\langle \mathcal{A}(e) \rangle$ denote the subdigraph of D induced by $\mathcal{A}(e)$. If D is 1-arc transitive, then the digraphs $\langle \mathcal{A}(e) \rangle$, for $e \in ED$, are all isomorphic to a fixed digraph, which will be denoted by $\Delta(D)$. If $\mathcal{A}$ is not the universal relation $ED \times ED$, then the structure of $\Delta(D)$ is quite restricted. (Recall that a *connected* digraph is one in which the underlying undirected graph is connected.)

Proposition 1.1. *Let D be a connected 1-arc transitive digraph. Then $\Delta(D)$ is 1-arc transitive and connected. Further, either*
(a) *$\mathcal{A}$ is the universal relation on ED and $\Delta(D) = D$, or*
(b) *$\Delta(D)$ is bipartite.*

(A digraph Δ is *bipartite* if the vertex set of Δ is the disjoint union $\Delta_1 \cup \Delta_2$ of two non-empty sets Δ_1 and Δ_2 such that the edge set of Δ is contained in $\Delta_1 \times \Delta_2$. The partition $V\Delta = \Delta_1 \cup \Delta_2$ is called a *bipartition* of Δ.)

This proposition will be proved in section 2. There are examples of highly arc transitive digraphs D for which the reachability relation $\mathcal{A}$ is universal. For example, if F is an ordered field, then the order digraph $D(F)$ of F with vertex set F and edge set $\{(a,b) | a < b\}$ has universal reachability relation, and the group of order-preserving permutations of F acts highly arc transitively on $D(F)$. (The last assertion is equivalent to the claim that the group of order-preserving permutations of F acts transitively on s-tuples in increasing order for all s — this is easily verified using piecewise linear permutations.) However, all such digraphs $D(F)$ have infinite in- and out-valencies. This motivates the following question. (Recall that a digraph D is said to be *locally finite* if for each $x \in VD$, the sets $D(x)$ and $D^*(x)$ are finite.)

Question 1.2. Are there any locally finite, highly arc transitive digraphs for which the reachability relation $\mathcal{A}$ is universal?

Let Δ be a fixed connected 1-arc transitive bipartite digraph. We shall be concerned with the class $\mathcal{D}(\Delta)$ of connected digraphs D for which the subdigraph $\langle \mathcal{A}(e) \rangle$ is isomorphic to Δ for each edge e of D. This class contains all connected 1-arc transitive digraphs D for which the associated digraph $\Delta(D)$ is isomorphic to Δ. A special member, $DL(\Delta)$, of $\mathcal{D}(\Delta)$ is defined in section 2, and is shown to be highly arc transitive and to be a covering digraph for each digraph in $\mathcal{D}(\Delta)$ (see Theorem 2.3). Moreover, if Δ is locally finite, then $DL(\Delta)$ is a universal covering digraph for $\mathcal{D}(\Delta)$ (see Theorem 2.6).

For most infinite highly arc transitive digraphs known to the authors, there is a digraph homomorphism onto the two-way infinite path Z. Moreover, it is shown in Section 3 that, if D is a connected 1-arc transitive digraph with a homomorphism onto Z, then the reachability relation $\mathcal{A}$ for D is not universal, and so $D \in \mathcal{D}(\Delta)$ for some connected, 1-arc transitive bipartite digraph Δ.

Question 1.3. Are there any highly arc transitive digraphs (apart from directed cycles) in $\mathcal{D}(\Delta)$, for some bipartite Δ, which do not have a digraph homomorphism onto Z?

As a partial answer to Question 1.3, we are able to show in Theorem 3.5 that all infinite highly arc transitive digraphs with restricted "spread" have a digraph homomorphism onto Z. (The same conclusion was also shown in [9] under the assumption that Δ is finite with unequal in- and out-valencies.)

In Section 4, several explicit constructions of highly arc transitive digraphs are given. The class investigated in Theorem 4.7 was the first new class of infinite highly arc transitive digraphs discovered by the authors. The class is analogous to a class of finite digraphs with high arc transitivity constructed by the second author in [8]. Further, this class of digraphs exhibits the type of behaviour which was investigated by the first author in [2]. In section 1 of [2], transitive permutation groups G having a pair of paired subconstituents which are "orthogonal" in a certain sense were investigated. For example, let G be transitive on Ω and, for $\alpha \in \Omega$, let $\Gamma(\alpha)$, $\Gamma^*(\alpha)$ be paired G_α-orbits such that either

(a) $G_\alpha^{\Gamma(\alpha)}$ and $G_\alpha^{\Gamma^*(\alpha)}$ have no nontrivial epimorphic image in common, or
(b) the sets $\Gamma(\alpha)$ and $\Gamma^*(\alpha)$ are finite of the same cardinality, and G_α is 2-transitive on one but not both of these sets.

Then the proof of Theorem 1.1 of [2] essentially shows that G acts highly arc transitively on the orbital digraph associated with Γ and Γ^*. In [3], it was remarked that this kind of orthogonality is possible for groups acting vertex transitively on trees (indeed, any two transitive permutation groups occur as paired subconstituents for such groups), and it was speculated that perhaps a highly arc transitive digraph was fairly close to being a tree. The class of highly arc transitive digraphs constructed in Theorem 4.7 shows that this speculation is wrong; and in fact any two transitive groups can occur as paired subconstituents in a group acting vertex transitively on a digraph of the type defined in Theorem 4.8 (see Remark 4.10(b)).

2. Universal covering digraphs

First, the proof of Proposition 1.1 is given.

Proof of Proposition 1.1. Let e', e'' be edges in $\mathscr{A}(e)$. Since D is 1-arc transitive, some automorphism g of D maps e' to e'', and so fixes setwise the equivalence class $\mathscr{A}(e) = \mathscr{A}(e') = \mathscr{A}(e'')$. It follows that the digraph $\langle \mathscr{A}(e) \rangle$ is 1-arc transitive. By definition $\langle \mathscr{A}(e) \rangle$ is connected.

Suppose that $\langle \mathscr{A}(e) \rangle$ contains a 2-arc (x_0, x_1, x_2). Then clearly each edge of D incident with x_1 is in $\mathscr{A}(e)$. Since D is connected and $\langle \mathscr{A}(e) \rangle$ is 1-arc transitive, $\mathscr{A}$ is universal.

If, on the other hand, $\langle \mathscr{A}(e) \rangle$ does not contain a 2-arc, then $\Delta(D) \cong \langle \mathscr{A}(e) \rangle$ is bipartite, with Δ_1 and Δ_2 the sets of vertices x such that (x,y) and (y,x) respectively are edges of Δ for some y. ∎

Let Δ be a connected bipartite digraph, with bipartition $V\Delta = \Delta_1 \cup \Delta_2$. Let $u = |\Delta_1|$ and $v = |\Delta_2|$, where u and v are not necessarily finite. Recall that $\mathscr{D}(\Delta)$ is the class of all connected digraphs D for which the subdigraph $\langle \mathscr{A}(e) \rangle$ is isomorphic to Δ for each edge e of D. A special digraph $DL(\Delta)$ in the class will now be defined. It is an edge induced subgraph of the line digraph of a tree.

Definition 2.1. Let T be the directed tree with constant in-valency u and constant out-valency v. At each vertex x of T, let ϕ_x be a bijection from $T(x) = \{y | (x,y) \in ET\}$ to Δ_2, and let ψ_x be a bijection from $T^*(x) = \{y | (y,x) \in ET\}$ to Δ_1. Then $DL(\Delta)$ is defined to be the digraph with vertex set ET such that, for $(w,x), (y,z) \in$

ET, $(w,x),(y,z)$ is an edge of $DL(\Delta)$ if and only if $x=y$ and $(\psi_x(w),\phi_y(z))$ is an edge of Δ.

One way to visualise this construction is to replace each vertex of T by a copy of Δ in such a way that the copies of Δ corresponding to vertices x and y of T, with $(x,y) \in ET$, have a single vertex e in common, namely a vertex in the set corresponding to Δ_2 for x and Δ_1 for y. If neither (x,y) nor (y,x) is an edge of T, then the copies of Δ corresponding to x and y are disjoint. Different choices of the bijections ϕ_x, ψ_x, for $x \in VT$, lead to isomorphic digraphs. This is not immediately obvious, and depends on the fact that T is a tree: we may inductively work out from a vertex along all branches of T, adjusting the bijections.

Recall that, for digraphs D and D', a *digraph homomorphism* $\phi: D \to D'$ is a map $\phi: VD \to VD'$ which maps edges of D to edges of D'. If ϕ is onto, then ϕ is called a digraph *epimorphism*, and D' is called an *epimorphic image* of D. If ϕ is a bijection, it is called a digraph *isomorphism*. Further, D is called a *covering digraph* of D' if there is a digraph epimorphism ϕ such that, for each vertex x of D, the restrictions of ϕ to $D(x)$ and $D^*(x)$ are isomorphisms from the digraphs induced on those sets to those induced on $D'(x\phi)$ and $D'^*(x\phi)$ respectively. The map ϕ is called a *covering projection* in this case.

Theorem 2.2. *Let Δ be a connected bipartite digraph, and let s be a positive integer. The digraph $DL(\Delta)$ belongs to the class $\mathcal{D}(\Delta)$. Moreover, if Δ is 1-arc transitive then, for any digraph $D' \in \mathcal{D}(\Delta)$, and any s-arcs a of $DL(\Delta)$ and a' of D', there is a covering projection $\phi: DL(\Delta) \to D'$ such that $a\phi = a'$.*

Proof. Let $D = DL(\Delta)$. By the definition of D, $\langle \mathcal{A}(e) \rangle \cong \Delta$ for each edge e of D. Also, the fact that D is connected follows from the connectivity of Δ and of the tree T. As we observed, the isomorphism type of D does not depend on the choices of the bijections ϕ_x and ψ_x.

Now assume that Δ is 1-arc transitive. For any digraph $D' \in \mathcal{D}(\Delta)$ and any edge $e \in ED'$, denote the vertex set of $\langle \mathcal{A}(e) \rangle$ by $V(e)$. (Recall that $\langle \mathcal{A}(e) \rangle$ is isomorphic to Δ.) If $e = (e_0, e_1)$ is an edge of D, where $e_0 = (w,x)$ and $e_1 = (x,y)$ are edges of T, we denote $V(e)$ also by $V(x)$, and denote the digraph $\langle \mathcal{A}(e) \rangle$ by $\mathcal{A}(x)$. As $E\Delta \subseteq \Delta_1 \times \Delta_2$, the vertices in Δ_1 and Δ_2 will be called *sources* and *sinks* respectively. Now let D' be a digraph in $\mathcal{D}(\Delta)$. Since Δ is 1-arc transitive, if e and e' are edges of D and D' respectively, then there is an isomorphism $\theta(e,e')$ from $\langle \mathcal{A}(e) \rangle$ to $\langle \mathcal{A}(e') \rangle$ which maps e to e'.

Let $a = (e_0, \ldots, e_s)$ and $a' = (f_0, \ldots, f_s)$ be s-arcs in D and D' respectively. A mapping ϕ from VD to VD' will be defined inductively. First define $\phi: V(e_0) \to V(f_0)$ as $\phi = \theta(e_0, f_0)$, and let $e_0 = (w, x_0)$. This mapping ϕ will be extended to all of VD inductively. At the $(k+1)^{\text{st}}$ step, for $k \geq 0$, the domain of ϕ will be the set V_k of all vertices of D which lie in $V(x)$ for some vertex x of T at distance at most k from x_0 (in the underlying undirected tree). We have already defined ϕ on V_0. Suppose then that $k \geq 1$ and that ϕ has been defined on V_{k-1}. Let x_k be a vertex of T at distance k from x_0. Let x_{k-1} be the vertex of T adjacent to x_k on the unique path $x_0, \ldots, x_{k-1}, x_k$ (in the underlying undirected tree) from x_0 to x_k, and let v_k denote the edge of T between x_{k-1} and x_k, so that v_k is (x_{k-1}, x_k) or (x_k, x_{k-1}). Then $v_k \in V(x_{k-1}) \subseteq V_{k-1}$, and so $v_k\phi$ has been defined already. Moreover, $V(x_k) \cap V_{k-1} = \{v_k\}$ (that is, v_k is the only element v of $V(x_k)$ for which $v\phi$ is defined so far), since each element of $V(x_k) \setminus \{v_k\}$ is an edge of T between

x_k and a vertex of T at distance $k+1$ from x_0. Now let e be an edge of D of the form (v_k, v) if $v_k = (x_{k-1}, x_k)$, or of the form (v, v_k) if $v_k = (x_k, x_{k-1})$. Moreover, if $k \leq s$ and the edge (e_{k-1}, e_k) is of this form, then choose $e = (e_{k-1}, e_k)$. Similarly, let e' be an edge of D' of the form $(v_k\phi, v'_k)$ if $v_k = (x_{k-1}, x_k)$ and of the form $(v'_k, v_k\phi)$ if $v_k = (x_k, x_{k-1})$, and choose $e' = (f_{k-1}, f_k)$ if $e = (e_{k-1}, e_k)$. Then we may define ϕ on $V(x_k)$ as $\theta(e, e')$. If x_k and x'_k are distinct vertices of T at distance k from x_0, then $V(x_k)$ and $V(x'_k)$ are disjoint. Thus, defining ϕ as above for all $x'_k \in V_k \setminus V_{k-1}$ yields a well-defined map from V_k to VD', completing the $(k+1)^{\text{st}}$ step of the definition of ϕ. Since D is connected, this procedure defines a function $\phi : VD \to VD'$; note that $a\phi = a'$.

First we check that ϕ is a digraph homomorphism. For any edge e of D, $\phi|_{V(e)}$ was defined as $\theta(\tilde{e}, e')$ for some $\tilde{e}$ in $\mathscr{A}(e)$ and some $e' \in ED'$, and so $e\phi$ is an edge of $\langle \mathscr{A}(e') \rangle$. Moreover, for each vertex $u = (x, y)$ of D, since the forward neighbourhood $D(u)$ is a subset of $V(y)$ and $\phi|_{\mathscr{A}(y)}$ is an isomorphism, it follows that $\phi|_{D(u)}$ is an isomorphism. Similarly, since $D^*(u) \subseteq V(x)$, $\phi|_{D^*(u)}$ is an isomorphism.

We claim that ϕ is an epimorphism, and hence a covering projection. Let $u \in V(x_0)$. Since $\phi|_{D(u)}$ and $\phi|_{D^*(u)}$ are isomorphisms, it follows that every vertex of D' at distance 1 (in the underlying graph of D) from $u\phi$ is in the image of ϕ. Thus, all vertices at distance at most 1 from an element of $V(x_0)\phi$ are in the image of ϕ. Suppose inductively that all vertices at distance at most k from a vertex in $V(x_0)\phi$ are in the image of ϕ, and let u' be a vertex at distance $k+1$ from a vertex in $V(x_0)\phi$. Then there is an edge e' of D' of the form (u', u'') or (u'', u'), such that u'' is distant at most k from an element of $V(x_0)\phi$. Hence $u'' = u\phi$ for some $u \in VD$. Since $\phi|_{D(u)}$ and $\phi|_{D^*(u)}$ are ismorphisms, u' must lie in the image of ϕ. By induction, ϕ is an epimorphism. Thus D is a covering digraph for D' with covering projection ϕ, and $a\phi = a'$, as required. ∎

Theorem 2.3. *Let Δ be a connected 1-arc transitive bipartite digraph. Then*
(a) *$DL(\Delta)$ is a highly arc transitive digraph, and*
(b) *$DL(\Delta)$ is a covering digraph for each digraph in $\mathscr{D}(\Delta)$.*

Proof. Taking $D' = DL(\Delta)$ in the proof of Theorem 2.2, the covering projection ϕ turns out to be an isomorphism. (For if $u\phi = v\phi$ for u, v distinct vertices of $DL(\Delta)$, then in the underlying graph of $DL(\Delta)$, each path from u to v is mapped by ϕ to a cycle containing $u\phi$. However, each such cycle lies in the underlying graph of $\langle \mathscr{A}(e) \rangle$ for some edge e, and hence there is an alternating path from u to v. Thus u and v are vertices of $\langle \mathscr{A}(e) \rangle$ for some edge e. Since $\phi|_{\langle \mathscr{A}(e) \rangle}$ is an isomorphism, it follows that $u\phi \neq v\phi$, which is a contradiction.) Thus for each $s \geq 1$, and for each pair a, a' of s-arcs of $DL(\Delta)$, there is an automorphism ϕ of $DL(\Delta)$ such that $a\phi = a'$. Thus (a) is proved. Part (b) has been proved already in Theorem 2.2 (with $s = 1$). ∎

Remark 2.4. If either $u = 1$ or $v = 1$ in Definition 2.1, then $E\Delta = \Delta_1 \times \Delta_2$ (that is, Δ is the complete bipartite digraph $K_{u,v}$), and $DL(\Delta)$ is the line digraph of T. Moreover, $DL(\Delta)$ is isomorphic to the tree T. On the other hand, if both u and v are greater than 1, then $DL(\Delta)$ is nothing like a tree, since the underlying graph of any $\langle \mathscr{A}(e) \rangle$ contains a cycle.

Now we show, for connected, locally finite, bipartite digraphs Δ, that $DL(\Delta)$ has a universal covering property with respect to $\mathscr{D}(\Delta)$.

Definition 2.5. (a) A covering digraph D of C is called a *universal covering digraph* of C if there is a covering projection $\phi: D \to C$ such that, for all covering digraphs D' of C and covering projections $\phi': D' \to C$, there exists a covering projection ψ: $D \to D'$ such that $\psi\phi' = \phi$.

(b) In any category, an object p is called *projective* if, for every morphism h: $p \to c$ and every epimorphism $g: b \to c$ in the category, there is a morphism $h': p \to b$ in the category such that $h'g = h$.

Theorem 2.6. *Let Δ be a connected, locally finite, 1-arc transitive, bipartite digraph. Then $DL(\Delta)$ is a projective object in the category whose objects are the digraphs in $\mathscr{D}(\Delta)$ and whose morphisms are the covering projections.*

Remark 2.7. Note that each covering projection is an epimorphism in the above category. Also, the conclusion of the theorem is much stronger than the statement that $DL(\Delta)$ is a universal covering digraph of each digraph in $\mathscr{D}(\Delta)$. The notion of a projective object as defined here is the usual one (see Maclane [7], p. 114); and the definition of a universal covering digraph is analogous to that of a universal covering space in topology (see Spanier [10], p. 80), but with some modifications required by the situation.

For the proof of Theorem 2.6, we will need the following lemma and corollary.

Lemma 2.8. *Let M be a connected, locally finite digraph which is either vertex transitive or 1-arc transitive, and let $\theta: M \to M$ be a covering projection. Then θ is an isomorphism.*

Proof. Suppose that $u\theta = v\theta$ for distinct vertices u and v of M. Since M is connected, there is a walk $(x_0, \ldots, x_k)$ in M with $x_0 = u$, $x_k = v$. (Recall that $(x_0, \ldots, x_k)$ is a *walk* if for each i $(1 < i < k)$, either (x_{i-1}, x_i) or (x_i, x_{i-1}) is an edge; the walk is *closed* or *open* according as $x_k = x_0$ or $x_k \neq x_0$ respectively.) Since θ maps closed walks to closed walks, the number of open walks of length k beginning at u is strictly greater than the number beginning at $u\theta$. If M is vertex transitive, then some automorphism of M maps u to $u\theta$, and we have a contradiction. Similarly, if M is 1-arc transitive, then, as θ induces digraph isomorphims from $M(u)$ to $M(u\theta)$ and from $M^*(u)$ to $M^*(u\theta)$, again some automorphism of M maps u to $u\theta$, and we have a contradiction. ∎

Corollary 2.9. *Let $C, C' \in \mathscr{D}(\Delta)$, where Δ is a connected, locally finite, 1-arc transitive, bipartite digraph, and let $\theta: C \to C'$ be a covering projection. Then, for all edges e of C, $\theta|_{\mathscr{A}(e)}$ is an isomorphism from $\langle \mathscr{A}(e) \rangle$ to $\langle \mathscr{A}(e\theta) \rangle$.*

Proof. For $e \in EC$, the subdigraph $\langle \mathscr{A} \rangle$ is isomorphic to the connected locally finite 1-arc transitive digraph Δ, and the restriction $\theta|_{\mathscr{A}(e)}$ is a covering projection from $\langle \mathscr{A}(e) \rangle$ onto $\langle \mathscr{A}(e\theta) \rangle$. Since $\langle \mathscr{A}(e\theta) \rangle$ is also isomorphic to Δ, it follows from Lemma 2.8 that $\theta|_{\mathscr{A}(e)}$ is an isomorphism. ∎

Proof of Theorem 2.6. Let $D = DL(\Delta)$, let $D', C \in \mathscr{D}(\Delta)$ and let $\phi: D \to C$ and ϕ': $D' \to C$ be covering projections. We define a covering projection $\psi: D \to D'$ such that $\psi\phi' = \phi$ as follows. Choose a "base" vertex $v_0 = (x_0, x_0')$ of D, and let $c_0 = v_0\phi \in VC$ and $c_0 = w_0\phi'$ for some $w_0 \in VD'$. Define $v_0\psi$ to be w_0. If e is any edge of D, say from (x', x) to (x, x'') (where x, x', x'' are vertices of T), let $\mathscr{A}(x)$ denote the subgraph $\langle \mathscr{A}(e) \rangle$. We shall extend the definition of ψ in a natural way.

By Corollary 2.9, $\phi|_{\mathscr{A}(x_0)}$ is an iscmorphism from $\mathscr{A}(x_0)$ to $\langle\mathscr{A}(f_0)\rangle$ for some edge $f_0=(c_0,c')$ in $\mathscr{A}(x_0)\phi$, and $\phi|_{\langle\mathscr{A}(g_0)\rangle}$ is an isomorphism from $\langle\mathscr{A}(g_0)\rangle$ to $\langle\mathscr{A}(f_0)\rangle$, where $g_0=(w_0,w')\in ED'$ and $g_0\phi'=f_0$. Define $\psi|_{\mathscr{A}(x_0)}$ to be $\phi|_{\mathscr{A}(x_0)}\left(\phi'|_{\langle\mathscr{A}(g_0)\rangle}\right)^{-1}$. Suppose inductively that ψ has been defined on each vertex v of D lying in $\mathscr{A}(x)$, for some x at distance at most k from x_0 in the underlying graph of the tree T, such that $v\phi=v\psi\phi'$. Let v be a vertex of $\mathscr{A}(x)$, where x lies on a path of length $k+1$ from x_0 (in the underlying graph of T, and hence not on any such path of length at most k): say $(x_0,\ldots,x_k,x_{k+1})$, with $x_{k+1}=x$. Then (x_k,x_{k+1}) or (x_{k+1},x_k) is an edge v' of T; so v' is a vertex of $\mathscr{A}(x_k)$, and $v'\psi$ has been defined and $v'\psi\phi'=v'\phi$. Further, v' is the only vertex of $\mathscr{A}(x)$ for which ψ has been defined so far. By Corollary 2.9, $\phi|_{\mathscr{A}(x)}$ is an isomorphism onto $\langle\mathscr{A}(f)\rangle$ for some edge $f=(v'\phi,c)$ of C, and $\phi'|_{\langle\mathscr{A}(g)\rangle}$ is an isomorphism from $\langle\mathscr{A}(g)\rangle$ to $\langle\mathscr{A}(f)\rangle$, where $g=(w',w)\in ED'$ and $g\phi=(w'\phi',w\phi')=f$. Define $\psi|_{\mathscr{A}(x)}$ to be $\phi|_{\mathscr{A}(x)}\left(\phi'|_{\langle\mathscr{A}(g)\rangle}\right)^{-1}$. Then $v\phi=v\psi\phi'$ for all $v\in\mathscr{A}(x)$. As in the proof of Theorem 2.2, since distinct vertices x,x' of T at distance $k+1$ from x_0 correspond to vertex disjoint digraphs $\mathscr{A}(x)$, $\mathscr{A}(x')$, this process yields a well defined map $\psi\colon VD\to VD'$. It follows from the definition of ψ that ψ is a digraph homomorphism, ψ is a local isomorphism, and $\phi=\psi\phi'$. Also, an argument similar to the one given in the proof of Theorem 2.2 shows that ψ is onto. Thus ψ is a covering projection and hence D is projective. In particular, D is a universal covering digraph for each digraph C of $\mathscr{D}(\Delta)$. Thus Theorem 2.6 is proved. ∎

The proof of Lemma 2.8 clearly uses the fact that M is vertex transitive or 1-arc transitive, and that M is locally finite.

Question 2.10. Is Lemma 2.8 still true if the hypothesis of vertex transitivity or 1-arc transitivity is deleted? That is, must a covering projection of a connected locally finite digraph be an isomorphism?

The following example shows that, at least for 1-arc transitive digraphs, local finiteness is a necessary condition for the conclusion of Lemma 2.8 to hold.

Example 2.11. Let w be an infinite cardinal. Let T be an infinite undirected tree with vertex set VT the disjoint union of three sets A,B and C such that, for each $a\in A$,

$$|T(a)\cap B|=|T(a)\cap C|=w, \qquad T(a)\cap A=\emptyset,$$

and, for each $b\in B$,

$$|T(b)\cap A|=4, \qquad T(b)\cap B=T(b)\cap C=\emptyset,$$

and, for each $c\in C$,

$$|T(c)\cap A|=8, \qquad T(c)\cap B=T(c)\cap C=\emptyset.$$

Clearly A,B,C are orbits of the automorphism group $\mathrm{Aut}(T)$ of T. Next (using the Axiom of Choice) define a map g from the edge set ET of T onto the set $I=\{1,2,\ldots,8\}=\mathbb{Z}_8$ such that, for each $b\in B$, the restriction of g to $T(b)$ is a 1–1 map onto $\{1,2,3,4\}$, and for each $c\in C$, the restriction of g to $T(c)$ is a bijection.

Let R be the digraph with vertex set $VR=A\cup A'$, the disjoint union of two copies of the set A, where we write $A'=\{a'|a\in A\}$, and with edge set $\{(a,a')|a\in A\}$.

We shall define a connected 1-arc transitive quotient digraph $\bar{R}$ of R, and a covering projection $\theta : \bar{R} \to \bar{R}$ which is onto but not 1–1. The digraph $\bar{R}$ will have infinite in- and out-valencies, thus demonstrating that the assumption of local finiteness in Lemma 2.5 is necessary, at least for 1-arc transitive digraphs. The quotient digraph $\bar{R}$ is defined in terms of two equivalence relations ρ and ρ' on A: the vertex set of $\bar{R}$ will be the disjoint union $A/\rho \cup A'/\rho'$, where A/ρ and A'/ρ' are the sets of equivalence classes of ρ and ρ' respectively. (We identify the relation ρ' on A with its image on A'). Furthermore, $(\rho(a_1), \rho'(a_2'))$ will be an edge of $\bar{R}$ if and only if there is an edge in R from a point of $\rho(a_1)$ to a point of $\rho'(a_2')$, that is, if and only if $\rho(a_1) \cap \rho'(a_2) \neq \emptyset$. (If $a \in \rho(a_1) \cap \rho'(a_2)$, then $(\rho(a_1), \rho'(a_2')) = (\rho(a), \rho'(a'))$.)

Two points of A will be equivalent modulo ρ (respectively ρ') if and only if they are in the same connected component of the following graph S (respectively S'): the graph S (respectively S') has vertex set A, and two points a_1, a_2 of A are adjacent in S (respectively S') if there is a path (a_1, x, a_2) of length 2 from a_1 to a_2 in T such that $\{g(\{a_1, x\}), g(\{a_2, x\})\}$ is $\{1,2\}$ or $\{3,4\}$ if $x \in B$, and is $\{2i-1, 2i\}$ for some i if $x \in C$ (respectively, is $\{2,3\}$ or $\{4,1\}$ if $x \in B$, and is $\{2i, 2i+1\}$ for some i if $x \in C$).

Thus each element of B, C corresponds to an alternating cycle of length 4,8 respectively in $\bar{R}$, and an edge $(\rho(a_1), \rho'(a_2'))$ of $\bar{R}$ lies in w alternating cycles of length 4, and w alternating cycles of length 8.

It can be readily checked that $\bar{R}$ is connected and has infinite in-valency and out-valency. Also R is 1-arc transitive, but verifying this is more difficult. We give a few details:

Any map f from $\{1,2\}$ or $\{3,4\}$ to $\{1,2,3,4\}$ with image $\{1,2\}$ or $\{3,4\}$ can be extended to a permutation of $\{1,2,3,4\}$ which preserves the partitions $\{12|34\}$ and $\{23|41\}$. Also, any map f from $\{2j-1, 2j\}$ to I, for some j, with image $\{2i-1, 2i\}$, for some i, can be extended to a bijection of I which preserves the partitions $\{12|34|56|78\}$ and $\{23|45|67|81\}$. It follows that, for each path (a_1, x, a_2) of length 2 in T with $a_1, a_2 \in A$ such that $\{g(\{a_1, x\}), g(\{a_2, x\})\}$ is one of $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ or $\{7,8\}$, there is an automorphism σ of T which interchanges a_1 and a_2 and which induces automorphisms of S_1 and S_2. (σ can be defined inductively on the distance in T of points from x.) Thus σ induces an automorphism of $\bar{R}$ which interchanges $(\rho(a_1), \rho'(a_1'))$ and $(\rho(a_2), \rho'(a_2'))$. Note that $\rho(a_1) = \rho(a_2)$ but $\rho'(a_1) \neq \rho'(a_2)$. It follows inductively that the subgroup of $\mathrm{Aut}(\bar{R})$ fixing the vertex $\rho(a_1)$ is transitive on the set of forward neighbours of $\rho(a_1)$ in $\bar{R}$. Similarly, the subgroup of $\mathrm{Aut}(\bar{R})$ fixing $\rho'(a_1')$ is transitive on the set of backward neighbours of $\rho'(a_1')$ in $\bar{R}$. Thus $\bar{R}$ is 1-arc transitive.

To define a covering projection $\bar{R} \to \bar{R}$ which is not an isomorphism, choose a vertex $b_0 \in B$ and a vertex $c_0 \in C$. There is a digraph homomorphism $\theta : T \to T$ which maps c_0 to b_0 and is 2-to-1, mapping all other vertices of C to vertices of C and preserving B and A. Moreover, θ can be defined so that it induces a covering projection from $\bar{R}$ to itself: first the action on $T(c_0)$ can be defined so that the 8-cycle of $\bar{R}$ corresponding to c_0 is mapped onto the 4-cycle corresponding to b_0. In general, the action of θ on each vertex x of T at distance k from c_0 is defined so that it preserves the sets B and C, while the action of θ on $T(x)$ is defined to preserve adjacency in S and S' and hence to preserve the cycle in $\bar{R}$ corresponding to x, for each such x. This gives a double covering of $\bar{R}$ by itself.

Remark 2.12. Not only does the digraph $\bar{R}$ defined above show that local finiteness is necessary in Lemma 2.8, but it can also be used to demonstrate that local finiteness is necessary in Theorem 2.6. For, if $D = C = D' = DL(\bar{R})$, we can choose $\phi: D' \to C$ to be a double covering so that ϕ cannot factor through ϕ'. Thus, $DL(\bar{R})$ is not a projective object in the category with elements of $\mathscr{D}(\bar{R})$ as objects and covering projections as morphisms.

3. Property Z

In this section we investigate connected 1-arc transitive digraphs which have a digraph homomorphism onto the two-way infinite path Z.

Definition 3.1. (a) The digraph Z is defined as the digraph with vertex set the set $\mathbb{Z}$ of integers and edges the pairs $(n,n+1)$ for $n \in \mathbb{Z}$.

(b) A digraph D is said to have *property* Z if there is a digraph epimorphism $\phi: D \to Z$.

For connected digraphs D, property Z is equivalent to the assertion that any cycle in the underlying graph of D has equally many forward and backward edges. This property holds for trees, integer lattices, etc.

First we prove some simple facts.

Lemma 3.2. (a) *For each connected, 1-arc transitive, bipartite digraph Δ, the digraph $DL(\Delta)$ has property Z.*

(b) *Let D be a connected 1-arc transitive digraph. If the reachability relation $\mathscr{A}$ for D is universal then D does not have property Z.*

Proof. (a) Let x_0 be a "base" vertex of the underlying tree T of $DL(\Delta)$. We define a function $h: VT \to Z$ as follows. Set $h(x_0)=0$. If $x \in VT \setminus \{x_0\}$, then x lies on a unique path from x_0 in the underlying undirected tree, say $(x_0,x_1,\ldots,x_k)$ with $x_k = x$ for some $k \geq 1$. The value $h(x_k)$ is defined inductively on k by the rule that $h(x_k) = h(x_{k-1}) + \delta$, where $\delta = 1$ if $(x_{k-1},x_k) \in ET$ and $\delta = -1$ if $(x_k,x_{k-1}) \in ET$. Then, for a vertex $e = (x,y)$ of $DL(\Delta)$, the function $e \mapsto h(y)$ defines a digraph epimorphism from $DL(\Delta)$ to Z.

(b) Suppose that $\mathscr{A}$ is the universal relation. If D has no 2-arcs, then D cannot have Z as a homomorphic image. So assume that D has a 2-arc (x,y,z). Since $\mathscr{A}$ is universal, there is an alternating walk $(x_0,\ldots,x_k)$ with (x_0,x_1) or (x_1,x_0) equal to (x,y), and (x_{k-1},x_k) or (x_k,x_{k-1}) equal to (y,z). Suppose that D has property Z. Then there is a map $\phi: VD \to \mathbb{Z}$ such that, for $e=(u,v)$ an edge of D, $v\phi = u\phi + 1$. Then $z\phi = x\phi + 2$, as (x,y,z) is a 2-arc; but, because of the alternating walk, $z\phi = x\phi \pm 1$, a contradiction. ∎

By Proposition 1.1 and Lemma 3.2, a connected 1-arc transitive digraph D with property Z lies in $\mathscr{D}(\Delta)$ for some Δ, and the digraph $DL(\Delta)$ in $\mathscr{D}(\Delta)$ has property Z. It is an open question whether, in general, "most" infinite digraphs in $\mathscr{D}(\Delta)$, and in particular whether all infinite highly arc transitive digraphs in $\mathscr{D}(\Delta)$, have property Z (see the Questions in the Introduction). Theorem 3.5 below is a partial answer to these questions. In the next section, several more examples of highly arc transitive digraphs will be given, all with property Z. In the trivial case where Δ is a path of length 1, the digraph $DL(\Delta)$ is isomorphic to Z, but the finite connected graphs in $\mathscr{D}(\Delta)$ are the directed cycles, which are highly arc transitive but do not have property Z. We make the following conjecture.

Conjecture 3.3. *If D is a connected highly arc transitive digraph in $\mathcal{D}(\Delta)$ with property Z, and $\phi : D \to Z$ is a digraph epimorphism such that the inverse image $0\phi^{-1}$ of 0 is finite, then Δ is a complete bipartite digraph (that is, $E\Delta = \Delta_1 \times \Delta_2$.)*

(Note that the finiteness of $0\phi^{-1}$ is necessary, in view of examples such as the digraphs $DL(\Delta)$.)

Remark 3.4. A non-trivial digraph satisfying the hypotheses of Conjecture 3.3 was found by Brendan McKay and the second author: the vertex set is $Z \times X^n$, and an edge goes from (i,x,y) to $(i+1,z,x)$ if $x \in X^{n-1}$. It can be easily verified that this digraph is highly arc-transitive with property Z; it is related to the construction in Proposition 4.7. It demonstrates that Conjecture 3.3 would become false if its conclusion were strengthened by adding the statement that $|\Delta_1| = |0\phi^{-1}|$.

The result below gives a sufficient condition for a digraph to have property Z.

Definition 3.5. Let D be a vertex-transitive digraph with finite out-valency, and let x be a vertex of D. For each positive integer k, let p_k be the number of distinct vertices y such that there exists a k-arc $(x_0, \ldots, x_k)$ with $x_0 = x$ and $x_k = y$. Then the *out-spread* of D is defined to be

$$\limsup_{k \to \infty} (p_k)^{1/k}.$$

The *in-spread* of a vertex-transitive digraph with finite in-valency is defined similarly.

Theorem 3.6. *Let D be a connected infinite highly arc transitive digraph with finite out-valency (respectively, in-valency) such that the out-spread (respectively, in-spread) of D is* 1. *Then D has property Z.*

Remark 3.7. If D is a tree with constant out-valency, then the outspread of D is equal to its out-valency. If $D = DL(\Delta)$, then the out-spread of D is equal to the out-valency of Δ. Similarly for in-spread and in-valency. Since these digraphs have property Z, the converse of Theorem 3.6 is certainly not true. Similar remarks apply to the highly arc transitive digraphs of Theorem 4.8.

Question 3.8. Given a real number $c > 1$, are there any highly arc transitive digraphs with out-spread or in-spread c which do not have property Z?

Let D be as in Theorem 3.6. By Lemma 3.2 and Proposition 1.1, $D \in \mathcal{D}(\Delta)$ for some Δ with finite out-valency (respectively, in-valency), and there is a digraph epimorphism $\phi : D \to Z$ such that, for each $(u,v) \in ED$, $v\phi = u\phi + 1$.

Question 3.9. For such a digraph D, is it true that the inverse image $0\phi^{-1}$ of 0 is finite?

Proof of Theorem 3.6. Let $t_1, \ldots, t_{2l}$ be non-negative integers with $t_i > 0$ for $1 < i < 2l$, where $l \geq 1$, and let $t = \sum t_i$. Let $W(t_1, \ldots, t_{2l})$ be the set of walks $w = (w_0, \ldots, w_{2l})$ in the underlying graph of D such that the first t_1 arcs (w_{i-1}, w_i) are "forward arcs" in D, that is, $(w_{i-1}, w_i) \in ED$; the next t_2 arcs are "backward arcs", that is, $(w_i, w_{i-1}) \in ED$; and so on. We *claim* that

If there is a closed walk $w \in W(t_1, \ldots, t_{2l})$, that is, one with $w_0 = w_{2l}$, then $\sum (-1)^i t_i = 0$.

This claim implies the result, by the following argument: Let v be a fixed vertex in D and set $v\phi = 0$. For each vertex x of D, there is a walk $w = (w_0, \ldots, w_k)$ such

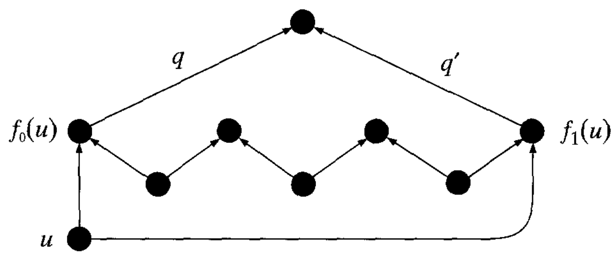

Fig. 1

that $w_0 = v$ and $w_k = x$; and $w \in W(t_1, \ldots, t_{2l})$ for some $t_1, \ldots, t_{2l}$ (where t_1 or t_{2l} may be zero). Define $x\phi$ to be $\sum(-1)^i t_i$. The fact that ϕ is well defined will follow from the proof of the claim. By definition, if $(x,y) \in ED$, then $y\phi = x\phi + 1$; so ϕ is a digraph homomorphism from D to Z. Also ϕ is an epimorphism: for let $w = (w_0, \ldots, w_k)$ be a walk with $w_0 = v$, $w_k = x$, and $x\phi = i$; then $u\phi = i+1$ for $u \in D(w_k)$, and $u\phi = i-1$ for $u \in D^*(w_k)$. Thus, D has property Z.

The claim above is proved by induction on l. The case $l = 1$ follows from Proposition 3.10 below (which will be proved at the conclusion of this proof).

Proposition 3.10. *Let D be a connected highly arc transitive digraph with finite out-valency. If $x, y \in VD$ are such that there is a directed path of length d from x to y, then either all directed paths from x to y have length d, or D is a directed cycle.*

Proof of Claim. Assume then that $l \geq 2$ and that the claim is true for $l-1$. We shall deduce the claim for l in the case where the out-spread is 1. The case of in-spread 1 is similar. Let $w = (w_0, \ldots, w_t)$ be a closed walk in $W(t_1, \ldots, t_{2l})$ for some $t_1, \ldots, t_{2l}$. If $t_1 = 0$, then some cyclic permutation $(w_i, \ldots, w_t, w_1, \ldots, w_i)$ of w is a closed walk in $W(t'_1, \ldots, t'_{2l-2})$, and by induction $\sum(-1)^i t_i = \sum(-1)^i t'_i = 0$. So we may suppose that $t_1 \neq 0$, and similarly that $t_{2l} \neq 0$. Since D is vertex-transitive, for each vertex u of D, $W(t_1, \ldots, t_{2l})$ contains a closed walk $w(u) = (w_0(u), \ldots, w_t(u))$, with $w_0(u) = w_t(u) = u$. With this notation, set $f_0(u) = w_{t_1}(u)$ and $f_1(u) = w_{t-t_{2l}}(u)$; thus $w(u)$ begins with a directed path of length t_1 from u to $f_0(u)$, and ends with a directed path of length t_{2l} from $f_1(u)$ to u (traversed in the reverse direction). Now suppose that, for some $k > 0$, some vertex v is such that

$$v = f_{i_k}(f_{i_{k-1}}(\cdots f_{i_1}(u)\ldots)) = f_{i'_k}(f_{i'_{k-1}}(\cdots f_{i'_1}(u)\ldots)),$$

for two different sequences $(i_1, \ldots, i_k)$ and $(i'_1, \ldots, i'_k)$ of zeros and ones, and some vertex u of D. Then there are directed paths p and p' from u to v via $f_{i_1}(u)$, $f_{i_2}(f_{i_1}(u))$ etc., and via $f_{i'_1}(u)$, $f_{i'_2}(f_{i'_1}(u))$, etc. respectively. Since $(i_1, \ldots, i_k) \neq (i'_1, \ldots, i'_k)$, we may assume without loss of generality that $i_1 = 0$ and $i'_1 = 1$, so that the path p goes via $f_0(u)$ and the path p' goes via $f_1(u)$. Let q, q' denote the parts of p, p' from $f_0(u)$ to v, $f_1(u)$ to v respectively. This is represented schematically in Figure 1.

If in the walk $w(u)$ the path from u to $f_0(u)$ is replaced by the path q (in reverse) from v to $f_0(u)$, and the path from $f_1(u)$ to u (in reverse) is replaced by the path q' from $f_1(u)$ to v, then a closed walk w' in

$$W(0, t_2 + t(q), t_3, \ldots, t_{2l-2}, t_{2l-1} + t(q'), 0),$$

beginning at v, is obtained, where $t(q), t(q')$ are the lengths of q, q' respectively. A cyclic permutation of w' is in

$$W(t_3, \ldots, t_{2l-2}, t_{2l-1} + t(q'), t_2 + t(q));$$

hence, by induction,

$$\sum_{i=2}^{2l-1} (-1)^i t_i + t(q) - t(q') = 0.$$

By Proposition 3.6, the two directed paths have equal length, that is, $t_1 + t(q) = t_{2l} + t(q')$. It follows that

$$\sum_{i=1}^{2l} (-1)^i t_i = 0.$$

The proof of the claim has thus been reduced to showing that such a vertex v exists. The existence of such a v follows from the fact that the out-spread is 1. For, if no such v existed, then for each k there would be 2^k distinct vertices of the form $f_{i_k}(f_{i_{k-1}}(\cdots f_1(u)\ldots))$. Thus, if M and m are the larger and smaller of t_1 and t_{2l}, then

$$\sum_{i=mk}^{Mk} p_i \geq 2^k,$$

and the out-spread of D would be greater than 1. ∎

Proof of Proposition 3.10. Suppose that there are vertices x, z_1 of D, and a positive integer d such that there are directed paths $p_1 = (x_0, \ldots, x_d)$ and $q_1 = (y_0, \ldots, y_k)$ with $x_0 = y_0 = x$ and $x_d = y_k = z_1$, where $k > d$. As D is d-arc transitive, there is an automorphism g_1 of D which maps $(y_0, \ldots, y_d)$ to p_1, hence maps q_1 to

$$q_2 = (x_0, \ldots, x_d, y_{d+1}^{g_1}, \ldots, y_k^{g_1}).$$

Let $z_2 = y_k^{g_1} = z_1^{g_1}$. Then $p_2 = p_1^{g_1}$ is a directed path of length d from x to z_2. Similarly, there is an automorphism g_2 of D which maps p_1 to p_2 and q_2 to the concatenation q_3 of p_2 and a $(k-d)$-arc (say r_3) from z_2 to $z_3 = z_2^{g_2}$, see Figure 2. Continuing in this way, we obtain a sequence of d-arcs $p_1, p_2, \ldots$ from x to points $z_1, z_2, \ldots$, and a sequence of k-arcs $q_1, q_2, \ldots$ from x to $z_1, z_2, \ldots$, such that, for each $i \geq 2$, q_i is the concatenation of p_{i-1} and a $(k-d)$-arc r_i from z_{i-1} to z_i.

Since D has finite out-valency, there is only a finite number of vertices which are end-points of d-arcs beginning at x. Thus, for some $i < j$, we must have $z_i = z_j$. Then the concatenation of $r_{i+1}, \ldots, r_j$ is a directed cycle in D of length $(k-d)(j-i)$. However, a highly arc transitive digraph with finite out-valency containing a (finite) directed cycle is finite, and in fact is a directed cycle. ∎

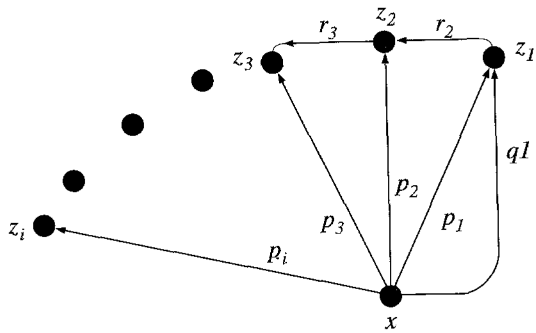

Fig. 2

4. More constructions of highly arc transitive digraphs

We present a few methods of obtaining new highly arc transitive digraphs from a given one.

Lemma 4.1. *Let D be a connected highly arc transitive digraph. Then:*

(a) *The line digraph of D is highly arc transitive and connected.*

(b) *For each $k \geq 1$, the digraph D_k, with vertex set VD and edges all pairs (x,y) of vertices for which there is a k-arc in D from x to y, is highly arc transitive but may not be connected.*

(c) *For each $s \geq 1$ and $k \geq 2s$, the digraph $D(s,k)$ with vertex set the set of s-arcs of D and edges all pairs (x,y) of s-arcs for which there is a k-arc $z=(z_0,\ldots,z_k)$ such that $x=(z_0,\ldots,z_s)$ and $y=(z_{k-s},\ldots,z_k)$, is highly arc transitive. However, $D(s,k)$ may not be connected.*

Proof. (a) The line digraph of D has vertex set ED, and (e,e') is an edge, for $e=(x,y)$ and $e'=(x',y')$ in ED, if and only if $y=x'$. Then for each $s\geq 1$, there is a 1–1 correspondence between s-arcs in the line digraph of D and $(s+1)$-arcs in D, which is preserved by the automorphism group of D, namely, $(e_0,\ldots,e_s)$ corresponds to $(x_0,\ldots,x_{s+1})$ if $e_i=(x_i,x_{i+1})$ for $i=0,\ldots,s$. Thus the automorphism group of D is transitive on the set of s-arcs of the line digraph of D for all $s\geq 1$.

(b) There is a 1–1 correspondence between the set of s-arcs of D_k and the set of sk-arcs of D which is preserved by automorphisms of D. To see that D_k may not be connected note that, if D has property Z, then the connected component of D_k containing a vertex x with $x\phi=n$ is contained in the set of vertices $\{y|y\phi\equiv n \pmod k\}$.

(c) Let $t \geq 1$ and let $x = (x^0,\ldots,x^t)$ be a t-arc in $D(s,k)$. By definition of adjacency in $D(s,k)$, for each i $(1 \leq i \leq t)$ there is a k-arc $z^i = (z_{(i-1)(k-s)},\ldots,z_{i(k-s)+s})$ in D such that $x^{i-1} = (z_{(i-1)(k-s)},\ldots,z_{(i-1)(k-s)+s})$ and $x^i=(z_{i(k-s)},\ldots,z_{i(k-s)+s})$. Then $z=(z_0,\ldots,z_{t(k-s)+s})$ is a $(t(k-s)+s)$-arc in D. Let $y=(y^1,\ldots,y^t)$ be another t-arc in $D(s,k)$, and let $w=(w_0,\ldots,w_{t(k-s)+s})$ be a $(t(k-s)+s)$-arc in D corresponding to y in the same way that z corresponds to x. Then there is an automorphism σ of D mapping z to w, as D is highly arc transitive, and clearly σ induces an automorphism of $D(s,k)$ mapping x to y. Thus $D(s,k)$ is t-arc transitive. ∎

To see that $D(s,k)$ may not be connected, note that if D has property Z, the connected component of $D(s,k)$ containing an arc $x=(x_0,\dots,x_s)$ with $x_0\phi=n$ is contained in the set of s-arcs $y=(y_0,\dots,y_s)$ such that $y_0\phi\equiv n \pmod{k-s}$.

It is not clear how large a class of highly arc transitive digraphs these constructions will produce. Note that if $D=DL(\Delta)$, then $D(s,k)$ is infinite even if Δ is finite. More flexible constructions are achieved using certain products of digraphs.

Definition 4.2. The *tensor product* $C\otimes D$ of two digraphs C and D is the digraph with vertex set $VC\times VD$, and with $((x,y),(x',y'))$ an edge in $C\otimes D$ if and only if $(x,x')\in EC$ and $(y,y')\in ED$.

The name tensor product is used since an adjacency matrix for $C\otimes D$ is a tensor or Kronecker product of adjacency matrices for C and D. This digraph is also called the *conjunction* of C and D (see [1]). Clearly $C\otimes D$ and $D\otimes C$ are isomorphic, and $C\otimes D$ is connected if and only if both C and D are connected.

Lemma 4.3. (a) *If C and D are both s-arc transitive digraphs, for some $s\geq 1$, then $C\otimes D$ is s-arc transitive. Thus, if both C and D are highly arc transitive then so is $C\otimes D$.*

(b) *If, for some $s\geq 1$, C is an s-arc transitive digraph with property Z, and D is a digraph such that $Z\otimes D$ is s-arc transitive, then $C\otimes D$ is s-arc transitive and has property Z. The same holds with "highly arc transitive" replacing "s-arc transitive".*

Proof. (a) The sequence $w=(w_0,\dots,w_s)$, where $w_i=(x_i,y_i)\in VC\times VD$, is an s-arc in $C\otimes D$ if and only if $x=(x_0,\dots,x_s)$ and $y=(y_0,\dots,y_s)$ are s-arcs in C and D respectively. Then, if C and D are s-arc transitive, it follows that the direct product $\mathrm{Aut}(C)\times\mathrm{Aut}(D)$ of the automorphism groups of C and D (with componentwise action) is transitive on the set of s-arcs of $C\otimes D$.

(b) Let $\phi:C\to Z$ be a digraph epimorphism. As in (a), an s-arc $w=(w_0,\dots,w_s)$ in $C\otimes D$, where $w_i=(x_i,y_i)\in VC\times VD$ for $0\leq i\leq s$, corresponds to an s-arc $x=(x_0,\dots,x_s)$ in C and an s-arc $y=(y_0,\dots,y_s)$ in D. Moreover, $z=(z_0,\dots,z_s)$, where $z_i=(x_i\phi,y_i)$ for $0\leq i\leq s$, is an s-arc in $Z\otimes D$. Now let w',x',y',z' be similarly-defined s-arcs in $C\otimes D$, C, D, $Z\otimes D$ respectively. Since C and $Z\otimes D$ are s-arc transitive, there are automorphisms ψ and θ of C and $Z\otimes D$ respectively such that $x\psi=x'$ and $z\theta=z'$. Denote the image of $(i,y)\in Z\otimes D$ under θ by $(i\theta_y,y\theta_i)$. Define the map $\eta:VC\times VD\to VC\times VD$ by

$$(x,y)\eta=(x\psi,y\theta_{(x\phi)}).$$

Then it is easily verified that η is a digraph homomorphism, and indeed is an automorphism of $C\otimes D$; and η maps w to w'. Thus $C\otimes D$ is s-arc transitive. Clearly the map $(x,y)\mapsto x\phi$ is a digraph epimorphism $C\otimes D\to Z$, so $C\otimes D$ has property Z. ∎

Remark 4.4. (a) In the tensor product construction of Lemma 4.3(a), the digraphs C, D could be taken as $DL(\Delta)$ for some Δ, or a related digraph, for example, the line digraph or the distance k digraph ($K\geq 2$) of $DL(\Delta)$ (see Lemma 4.1).

(b) For the construction given in Lemma 4.3(b), C could be taken as $DL(\Delta)$ for some Δ. One possibility for the digraph D is a complete digraph $K(\Sigma)$ (with loops) on a set Σ, that is, $VD=\Sigma$, $ED=\Sigma\times\Sigma$. For any digraph C, the tensor product $C\otimes K(\Sigma)$ is the digraph which results from replacing each edge of C by a copy of the complete bipartite digraph $K(\Sigma,\Sigma)$. If C is highly arc transitive then $C\otimes K(\Sigma)$ turns out to be highly arc transitive, whether or not C has property Z. The idea for this construction came from Bill Jackson [6].

Theorem 4.5. *Let C be a connected highly arc transitive digraph, and let Σ be a set. Then $C\otimes K(\Sigma)$ is a connected highly arc transitive digraph, and if C has property Z then $C\otimes K(\Sigma)$ also has property Z.*

Proof. Clearly $C\otimes K(\Sigma)$ is connected. To show that it is highly arc transitive, let s be a positive integer, and let $x=(x_0,\ldots,x_s)$ and $x'=(x'_0,\ldots,x'_s)$ be two s-arcs in $C\otimes K(\Sigma)$. Let $x_i=(c_i,u_i)$ and $x'=(c'_i,u'_i)$, where $c_i,c'_i\in VC$ and $u_i,u'_i\in\Sigma$, and $(c_i,c_{i+1}),(c'_i,c'_{i+1})\in EC$ for eact i. Now, for each automorphism σ of C, the map $(c,u)\mapsto(c^\sigma,u)$, for $(c,u)\in VC\times\Sigma$, is an automorphism of $C\otimes K(\Sigma)$, and as C is highly arc transitive, we may assume that $c_i=c'_i$ for all i. Also, for each permutation τ of Σ and for each $c'\in VC$, the map $\tau_{c'}$ which fixes all points (c,u) with $c\neq c'$ and maps (c',u) to (c',u^τ) is an automorphism of $C\otimes K(\Sigma)$. An appropriate sequence of these maps will take x to x'. Thus $C\otimes K(\Sigma)$ is s-arc transitive and hence highly arc transitive. Finally, if $\phi: C\to Z$ is a digraph epimorphism, then the map $(c,u)\mapsto c\phi$ is a digraph epimorphism from $C\otimes K(\Sigma)$ to Z. ∎

The original idea for the construction of Lemma 4.3(b) came from a particular class of highly arc transitive digraphs related to the sequences digraphs discussed below.

Definition 4.6. Let Δ be a connected bipartite digraph with bipartition $V\Delta=\Delta_1\cup\Delta_2$, and let δ_1,δ_2 be (fixed) elements of Δ_1,Δ_2 respectively. The *sequences digraph $S=S(\Delta;\delta_1,\delta_2)$ for Δ based at δ_1,δ_2* is defined as follows. The vertex set VS is the set of all (two-way infinite sequences) $x\in(V\Delta)^{\mathbb{Z}}$ such that

$x_n\in\Delta_1$ for all $n<0$;
$x_n=\delta_1$ for all but finitely many $n<0$;
$x_n\in\Delta_2$ for all $n\geq 0$;
$x_n=\delta_2$ for all but finitely many $n\geq 0$.

The pair (x,y) is an edge of $S(\Delta;\delta_1,\delta_2)$ if and only if

$y_i=x_{i-1}$ for all $i\neq 0$;
$(x_{-1},y_0)\in E\Delta$.

If the base points δ_1,δ_2 are clear from the context, then $S(\Delta;\delta_1,\delta_2)$ may be denoted by $S(\Delta)$, even though different choices of δ_1,δ_2 may lead to non-isomorphic sequences digraphs. The *zero sequence* of $S(\Delta)$ is the sequence 0 such that $0_n=\delta_1$ if $n<0$, and $0_n=\delta_2$ if $n\geq 0$.

The digraphs $S(\Delta;\delta_1,\delta_2)$ and $S(\Delta;\delta'_1,\delta'_2)$ are isomorphic if the pairs (δ_1,δ_2) and (δ'_1,δ'_2) are in the same orbit of $\mathrm{Aut}(\Delta)$ on $\Delta_1\times\Delta_2$. In particular, if Δ is the complete bipartite digraph $K(\Delta_1,\Delta_2)$, then all choices of $\delta_1\in\Delta_1$ and $\delta_2\in\Delta_2$ lead to isomorphic digraphs. The digraphs $Z\otimes S(\Delta)$ with Δ complete bipartite were the first class of infinite highly arc transitive digraphs constructed by the authors in their work on this paper. They are natural analogues of the finite s-arc transitive digraphs constructed by the second author in [8], and are also related to the de Bruijn digraphs (see Hall [4], Chapter 9). The fact that they are highly arc transitive follows from the next result.

Proposition 4.7. *Let Δ be a connected 1-arc transitive bipartite digraph with bipartition $V\Delta=\Delta_1\cup\Delta_2$, and let A be the automorphism group of Δ. Let $\delta_1\in\Delta_1$ and $\delta_2\in\Delta_2$, and let $D=Z\otimes S(\Delta;\delta_1,\delta_2)$. For $\tau\in A$ and $j\in Z$, define $\tau^{(j)}:VD\to VD$ by*

$$\tau^{(j)}:(i,x)\mapsto(i,x^{(j)}),$$

where $x^{(j)}$ is the result of applying τ to the $(i-j)^{\text{th}}$ entry of x, that is,

$$x_k^{(j)}=\begin{cases}x_k & \text{if } k\neq i-j,\\ x_k\tau & \text{if } k=i-j.\end{cases}$$

Then $\tau^{(j)}$ is an automorphism of D. Also, the map ϕ defined by $\phi:(i,x)\mapsto(i+1,x)$ is an automorphism of D. The group G of automorphisms of D generated by ϕ and the $\tau^{(j)}$, for $\tau\in A$ and $j\in Z$, is isomorphic to the wreath product $A\operatorname{wr}\mathbb{Z}$, with the top group generated by ϕ and the bottom group having j^{th} coordinate subgroup $A_j=\{\tau^{(j)}|\tau\in A\}\cong A$. Further, this group G acts transitively on VD and on s-arcs of D for all $s\geq 1$.

Proof. It is easy to verify that ϕ is an automorphism, and straightforward but tedious to verify that, for all $\tau\in A$ and all $j\in Z$, $\tau^{(j)}$ is also an automorphism of D. Also the map $\tau\mapsto\tau^{(j)}$ defines a monomorphism from A into $\operatorname{Aut}(D)$ with image A_j. It follows from the definition that elements of A_j and A_k, for $k\neq j$, commute with each other. Thus $\langle A_j|j\in Z\rangle$ is the direct product of the A_j, and $G=\langle\phi,A_j|j\in Z\rangle$ is isomorphic to the wreath product $A\operatorname{wr}\mathbb{Z}$.

We claim that G is transitive on VD. Since D is connected, it is sufficient to show that, for each 1-arc (w_0,w_1) of D, there is an automorphism in G which maps w_0 to w_1. The vertices w_0,w_1 are of the form $w_0=(i,x)$, $w_1=(i+1,y)$, where $y_j=x_{j-1}$ for $j\neq 0$, and $(x_{-1},y_0)\in E\Delta$. There are integers $m<0$ and $n\geq 0$ such that

$x_k=\delta_1$ for $k\leq m$ and $x_{m+1}\neq\delta_1$, and

$x_k=\delta_2$ for $k\geq n$ and $x_{n-1}\neq\delta_2$.

Since Δ is 1-arc transitive and connected, A is transitive on Δ_1 and Δ_2, so for $m<k<n$ there is $\sigma_k\in A$ such that $x_k\sigma_k=y_k$. Then the automorphism

$$(\sigma_{m+1})^{(i-(m+1))}\dots(\sigma_n)^{(i-n)}\phi$$

maps w_1 to w_2. Thus G is transitive on VD.

We show that G is transitive on s-arcs by induction on $s\geq 1$. Suppose first that $s=1$ and that w,w' are two 1-arcs. Since G is transitive on VD, we may assume that $w=(w_0,w_1)$, $w'=(w_0,w_1')$ with $w_0=(0,0)$. Then $w_1=(1,x)$ and $w_1'=(1,y)$ where

$x_k=y_k=\delta_1$ for $k<0$,

$x_k=y_k=\delta_2$ for $k>0$,

$(\delta_1,x_0),(\delta_1,y_0)\in E\Delta$.

There is an element $\tau\in A$ which maps (δ_1,x_0) to (δ_1,y_0); then $\tau^{(1)}\in G$ maps w to w'.

Suppose inductively that $s\geq 2$ and G is transitive on $(s-1)$-arcs. Let w,w' be two s-arcs in D. By induction we may assume that $w=(w_0,\dots,w_{s-1},w_s)$, $w'=$

$(w_0,\dots,w_{s-1},w'_s)$, with $w_{s-1}=(0,0)$. Then $w_i=(i-s+1,x^i)$ for $0\le i\le s-1$, with $x^{s-1}=0$, and $w_s=(1,x)$, $w'_s=(1,y)$ for some $x_i,x,y\in S(\Delta)$. As above,

$x_k=y_k=\delta_1$ for $k<0$,

$x_k=y_k=\delta_2$ for $k>0$,

$(\delta_1,x_0),(\delta_1,y_0)\in E\Delta$.

Let $\sigma\in A$ map (δ_1,x_0) to (δ_1,y_0). Then $\sigma^{(1)}\in G$ maps w_s to w'_s, and fixes w_{s-1}. Further, by the definition of adjacency in D, $(x^i)_{i-s}=\delta_1$; as $\sigma^{(1)}$ acts on the $(i\text{–}s)^{\text{th}}$ entry of x^i in $w_i=(i-s+1,x^i)$, it follows that $\sigma^{(1)}$ fixes w_i for $0\le i\le s-1$, and so $\sigma^{(1)}$ sends w to w'. By induction, G is transitive on the set of s-arcs of D for all $s\ge 1$. ∎

We next show that $Z\otimes S(\Delta)$ is in $\mathscr{D}(\Delta)$. This yields, as a corollary, a construction of a far larger class of highly arc transitive digraphs.

Theorem 4.8. *Let Δ be a connected bipartite digraph with bipartition $V\Delta=\Delta_1\cup\Delta_2$, and let $\delta_1\in\Delta_1$, $\delta_2\in\Delta_2$. Then*

(a) *the sequences digraph $S(\Delta;\delta_1,\delta_2)$ is connected; and*

(b) *if Δ is 1-arc transitive then the tensor product $Z\otimes S(\Delta;\delta_1,\delta_2)$ is a connected highly arc transitive digraph in $\mathscr{D}(\Delta)$.*

Corollary 4.9. *If Δ is a connected 1-arc transitive bipartite digraph with bipartition $V\Delta=\Delta_1\cup\Delta_2$, and $\delta_1\in\Delta_1$, $\delta_2\in\Delta_2$, and if D is a connected highly arc transitive digraph with property Z, then $D\otimes S(\Delta;\delta_1,\delta_2)$ is a connected highly arc transitive digraph.*

Proof of Theorem 4.8. (a) For $x\in VS(\Delta)$, there are integers $m<0$ and $n\ge 0$ such that

$x_i=\delta_1$ for all $i\le m$ and $x_{m+1}\ne\delta_1$, and

$x_i=\delta_2$ for all $i\ge n$ and $x_{n-1}\ne\delta_2$.

Thus, the sequence $(x_{m+1},\dots,x_{n-1})$, of length $k(x)=n-m-1$, contains all entries in x different from δ_1 and δ_2. We prove by induction on $k(x)$ that there is a walk in the underlying undirected graph from x to 0. If $k(x)=0$, then $x=0$. So assume that $k\ge 1$, and that all vertices x with $k(x)\le k-1$ lie on walks to 0. Suppose that $k(x)=k=n-m-1$, with n,m as defined above. If $m<-1$, then there is a directed path $(w_0,w_1,\dots,w_{-m-1})$ in $S(\Delta)$, where $w_0=x$ and, for $1\le j\le -m-1$,

$(w_j)_i=(w_{j-1})_{i-1}$ if $i\ne 0$,

$((w_{j-1})_{-1},(w_j)_0)\in E\Delta$.

Then w_{-m-1} has i^{th} entry δ_1 for $i\le -1$, δ_2 for $i\ge k$. Thus we may assume that $m=-1$, that is, $x_i=\delta_1$ for all $i\le -1$. Now, since Δ is connected, there is a walk $(y_0,\dots,y_{2r+1})$ in Δ with $y_0=x_0$ and $y_{2r+1}=\delta_1$ for some $2r+1\ge 1$. There is a corresponding walk $(w_0,\dots,w_{2r+1})$ in $S(\Delta)$, where

$(w_{2j})_i$ is x_i if $i\ne 0$, and is y_{2j} if $i=0$, and

$(w_{2j+1})_i$ is x_{i+1} if $i\ne -1$ and is y_{2j+1} if $i=-1$,

for $0\le j\le r$. Then $(w_{2r+1})_i$ is δ_1 if $i\le -1$ and is δ_2 if $i\ge k-1$, so $k(w_{2r+1})\le k-1$. By induction, w_{2r+1}, and hence x, lies on a walk to 0. It follows that $S(\Delta)$ is connected.

(b) The digraph $D=Z\otimes S(\Delta)$ is connected, since Z and $S(\Delta)$ are connected. Also it is easily verified that, for each edge e of D, $\langle\mathscr{A}(e)\rangle=\Delta$, so $D\in\mathscr{D}(\Delta)$. Finally, D is s-arc transitive for each $s\ge 1$, by Proposition 4.7. ∎

Remark 4.10. (a) The tensor product operation on digraphs is associative, that is $(B\otimes C)\otimes D \cong B\otimes(C\otimes D)$, so iterating the construction of Lemma 4.3 gives no more examples of highly arc transitive digraphs than the ones obtained from one application of these tensor product constructions. This is true in a stronger sense for the special subclass of digraphs $C\otimes S(\Delta)$, where C is highly arc transitive with property Z, and Δ is connected and 1-arc transitive. This is because

$$(C\otimes S(\Delta))\otimes S(\Sigma)\cong C\otimes S(\Theta),$$

where if $V\Delta=\Delta_1\cup\Delta_2$ and $V\Sigma=\Sigma_1\cup\Sigma_2$ are the bipartitions of Δ and Σ, then

$$V\Theta=(\Delta_1\times\Sigma_1)\cup(\Delta_2\times\Sigma_2),$$

and Θ is the subgraph of $\Delta\otimes\Sigma$ induced on this set (containing all edges of $\Delta\otimes\Sigma$).

(b) The class of digraphs considered in Proposition 4.6 has the property that, for any two transitive permutation groups B_1 and B_2, there is a digraph in the class admitting a vertex transitive group of automorphisms with B_1 and B_2 as paired subconstituents. For let B_1, B_2 act on disjoint sets Δ_1, Δ_2 respectively, and let Δ be the complete bipartite digraph with bipartition $V\Delta=\Delta_1\cup\Delta_2$. Then the automorphism group A of Δ is the direct product of the symmetric groups on Δ_1 and Δ_2, and its subgroup $B=B_1\times B_2$ is 1-arc transitive on Δ. It follows from the proof of Proposition 4.7 that the subgroup H generated by ϕ and $\tau^{(j)}$, for $\tau\in B$ and $j\in Z$, is isomorphic to $B\operatorname{wr}Z$ and acts highly arc transitively on $D=Z\otimes S(\Delta)$. Moreover, H is a transitive permutation group on VD and, for $\alpha\in VD$, the permutation groups induced by H_α on $D(\alpha)$ and $D^*(\alpha)$ are permutationally isomorphic to B_1 and B_2 respectively, so that B_1 and B_2 are paired subconstituents for H.

(c) Note that $Z\otimes S(\Delta)$ is in general more highly connected than any $DL(\Delta')$, the digraphs $DL(\Delta')$ all having connectivity 1.

George Bergman [1] constructed an interesting family of infinite permutation groups, from which we constructed a family of highly arc transitive digraphs D.

Definition 4.11. Let u and v be relatively prime positive integers, and let A be the additive subgroup of all rational numbers of the form $w/u^m v^n$ for $w,m,n\in\mathbb{Z}$. The digraph $D(u,v)$ is defined to have vertex set $\mathbb{Z}\times A/\mathbb{Z}$ (which can be thought of as an infinite sequence of "circles" $A/\mathbb{Z}$), and edge set the orbit under the group G defined below containing $((0,0),(1,0))$. The group G has generating set $\{g,h\}$, where

$$g:(n,r)\mapsto(n+1,r),$$
$$h:(n,r)\mapsto(n,r+(u/v)^n).$$

It can be verified that G is the semidirect product of the normal closure H of h and $\langle g\rangle\cong\mathbb{Z}$, and $H=\{h^a|a\in A\}$, where $h^a:(n,r)\mapsto(n,r+a(u/v)^n)$. Further, $D(u,v)$ is connected and has out-valency v and in-valency u, since $G_{(0,0)}=\langle h\rangle$ and $G_{(1,0)}=\langle h^{v/u}\rangle$; and G acts highly arc transitively on $D(u,v)$. It turns out that the digraph $D(u,v)$ is isomorphic to $Z\otimes S(K_{u,v})$:

Theorem 4.12. *The digraph $D(u,v)$ is connected and is isomorphic to $Z\otimes S(K_{u,v})$, where u and v are relatively prime positive integers.*

Proof. Let $\Delta=K_{u,v}$ with $\Delta_1=\{0,1,\ldots,u-1\}$, $\Delta_2=\{0,1,\ldots,v-1\}$, and consider $S(\Delta)=S(\Delta;0,0)$. (As we remarked above, if Δ is complete bipartite, then the isomorphism type of $S(\Delta;\delta_1,\delta_2)$ is independent of the choice of (δ_1,δ_2).) Define a map $\psi: S(\Delta)\to A/\mathbb{Z}$ by

$$x\mapsto \sum x_j(u/v)^{j+\epsilon(j)} \pmod{\mathbb{Z}},$$

where $\epsilon(j)$ is 0 if $j<0$ and 1 if $j\geq 0$. First, we show that ψ is 1–1. Suppose that $x\psi=y\psi$. By the definition of $S(\Delta)$, there is a positive integer k such that $x_i=y_i$ for all i with $|i|>k$. Now we have to show that, if c_i are integers such that

$0\leq c_i\leq u-1$ for $i<k$,

$0\leq c_i\leq v-1$ for $i>k$, and

$$\sum_{i=0}^{2k+1} c_i u^i v^{2k+1-i}\equiv 0 \pmod{u^k v^{k+1}},$$

then $c_i=0$ for all $i\neq k$.

This congruence implies that u^k divides $\sum_{i=0}^{k-1} c_i u^i v^{2k+1=i}$, from which we deduce (by induction on i) that $c_i v^{2k+1-i}\equiv 0 \pmod{u}$. Since u and v are relatively prime, it follows that $c_i=0$ for $0\leq i\leq k-1$. Similarly we find that $c_i=0$ for $k+1\leq i\leq 2k+1$. Thus ψ is 1–1.

Next we show that ψ is onto, that is, we show that, modulo $\mathbb{Z}$, each number of the form $w/u^m v^n$, for $w,m,n\in\mathbb{Z}$, is in the image of ψ. Since $0\psi=0$, we may assume that $m\geq 0$, $n\geq 0$, and $m+n>0$.

If m and n are both positive then, since u and v are coprime, there are integers a and b such that $au^m+bv^n=1$, and hence $w/u^m v^n=(wa/v^n)+(wb/u^m)$. We show by induction on k that, for all $m\geq 0$ and $n\geq 0$ with $m+n=k$, and for all integers a and b, there exists $x\in VS(\Delta)$ with

$x_i=0$ for $i<-m$ and for $i\geq n$, and

$x\psi\equiv(a/v^n)+(b/w^m) \pmod{\mathbb{Z}}$.

If $k=0$, then $(a/v^n)+(b/u^m)\equiv 0=0\psi$ modulo $\mathbb{Z}$. Suppose, inductively, that $k>0$ and that the result is true for all m,n with $m+m\leq k-1$; let m,n be non-negative integers with $m+n=k$. Let a and b be integers. Suppose first that $n>0$. Then $a=a_0+a_1v$, where $0\leq a_0\leq v-1$. Choose x with $x_i=0$ for $i\neq n-1$. We have $x\psi=x_{n-1}u^n/v^n$ and, as u and v are coprime, x_{n-1} may be chosen so that $x_{n-1}u^n=a_0+cv$ for some integer c. Then $x\psi=(a_0/v^n)+(c/v^{n-1})$. By induction, there is a y with $y_i=0$ for $i<-m$ and for $i\geq n-1$ such that $y\psi=((a_1-c)/v^{n-1})+b/u^m$. Then $x+y\in VS(\Delta)$, $(x+y)_i=x_i+y_i=0$ for $i<-m$ and for $i\geq n$, and

$$(x+y)\psi=(a/v^n)+(b/u^m),$$

so we are done.

Thus we may assume that $n=0$. Then $m>0$ and we argue similarly to establish the result. We conclude that ψ is onto, and hence is a bijection.

Now define $\phi: V(Z\otimes S(\Delta))\to VD(u,v)$ by $\phi:(i,x)\mapsto(i,x\psi)$. Then ϕ is a bijection. Let e be an edge of $Z\otimes S(\Delta)$. Then $e=((i,x),(i+1,y))$ for some i,x,y where $y_j=x_{j-1}$ for $j\neq 0$; and $e\phi=((i,x\psi),(i+1,y\psi))$ will be an edge of $D(u,v)$ if and only if $e\phi g^{-i}=((0,x\psi),(1,y\psi))$ is an edge of $D(u,v)$, (where $g:(i,r)\mapsto(i+1,r)$, as defined above). To see that this is an edge, recall that $ED(u,v)$ is the G-orbit containing $((0,0),(1,0))$. Consider the elements $t_q=g^q h g^{-q}$ of G for integers q.

Now $(0,z\psi)t_q=(0,z'\psi)$ where
$z'=z$ if $q=0$,
$z'_j=z_j$ for $j\neq q-1$ and $z'_{q-1}=z_{q-1}+1 \pmod v$ if $q>0$, and
$z'_j=z_j$ for $j\neq q$ and $z'_q=z_q+1 \pmod u$ if $q<0$.
It follows that

$$((0,x\psi),(1,y\psi))\left(\prod_{q\geq 1} t_q^{-x_{q-1}}\right) t_0^{-y_0} \left(\prod_{q\leq -1} t_q^{-x_q}\right) = ((0,0),(1,0)).$$

Hence $e\psi$ is an edge of $D(u,v)$.

It follows that $Z\otimes S(\Delta)\cong D(u,v)$, and, in particular, $D(u,v)$ is connected. ∎

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